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Improved Warning Limits Control Chart to Detect Shifts in the Process Mean

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Abstract

For univariate processes, Rattihalli et al. (2021-2022) proposed 'Modified Control Chart with Warning Limits to monitor the process mean' ($MCCWL$-$\overline{X}$). Also, Gadre and Rattihalli (2022) introduced 'Improved Control Chart to monitor the process mean' (ICC-$\overline{X}$). Here, we propose 'Improved Warning Limits control chart to detect shift in the process mean' (IWL-$\overline{X}$) chart. It is numerically illustrated that, IWL-$\overline{X}$ chart significantly reduces the out of control 'Average Time to Signal' (ATS) as compared to the $\overline{X}$ chart, $MCCWL$-$\overline{X}$ and ICC-$\overline{X}$.

Key Words: Control Limits, Warning Limits, Average Run Length, Average Time to Signal, $MCCWL$-$\overline{X}$ and ICC-$\overline{X}$.

1. Introduction

Shewhart type control charts are widely used in industries to monitor quality of a production process. These charts are quite effective in detecting large shifts and are insensitive for small to moderate shifts in the process mean. In $\overline{X}$ chart, the decision about status of the process depends only on the most recent value of the sample mean. For univariate production processes, many researchers like Page (1954), Roberts (1959) have proposed control charts to monitor the process mean. At the beginning, 'Average Run Length' (ARL) model is used to develop the related charts. For various control charts, one may refer to Montgomery (1996).

For Shewhart type control charts, the production process is considered as in control, if the current value of the sample mean is within the control limits, irrespective of its departure from the central line. One may refer to Montgomery (1996).

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A sample point (value of the sample mean) being close to the control limits, though within the control limits, gives a stronger evidence to suspect the quality of the production process. Thus, it is desirable to consider the relative position of a point in assessing quality of the production process. This aspect is accounted in control charts with the warning limits. By using the relative position of a point, Mahadik (2013) developed Shewhart $\bar{X}$ chart with varying warning and control limits. While developing his control chart, the idea of variable warning limits and variable control limits been used. Li et al. (2014) proposed various methods of the computations of ARL and ATS. Some of them are Markov chain approach and integration equation method. Wu et al. (2001) introduced the term ATS.

Rattihalli et al. (2021-2022) introduced a 'Modified Control Chart with Warning Limits to detect shift in the process mean' ($MCCWL\text{-}\overline{X}$). They used the Markov chain approach to derive ARL and hence the ATS expressions of the $MCCWL\text{-}\overline{X}$. In $MCCWL\text{-}\overline{X}$, it assumed that, at time zero, value of the sample mean $\overline{X} \in A$ (Acceptance Region). Also, Gadre and Rattihalli (2022) developed 'Improved Control Chart to detect shift in the process mean' **(**$ICC\text{-}\overline{X}$).

It is to be noted that, though the sample mean $\overline{X} \in A$, $\overline{X}$ may be much close to the control limits. In such a case, it assumed that, at time zero, $\overline{X} \in W$ (Warning Region). By taking into consideration of this fact, a control chart called the 'Improved Warning Limits control chart to detect shift in the process mean' ($IWL\text{-}\overline{X}$) chart is proposed. Let 'R' be the rejection region, of $IWL\text{-}\overline{X}$ chart. Under the above mentioned assumption, the stopping rule for $IWL\text{-}\overline{X}$ chart is 'Declare the process as out of control, if $\overline{X} \in R$ or two successive sample means fall in W'.

Remainder of this article is organized as follows. **Section-2** covers brief review of the control charts. **Section-3** covers the basic notations essential to design the proposed chart. The operation and the ATS expression of $IWL\text{-}\overline{X}$ chart are also given in the same section. Some numerical illustrations and comparative study of $IWL - \bar{X}$ chart with the $\bar{X}$ chart and $MCCWL - \bar{X}$ is carried out in **Section-4**. Concluding remarks are given in **Section-5**.

2. Brief Review of the control charts with warning limits:

2.1. Variable Control and Warning Limits to monitor the process mean ($VCWL\text{-}\overline{X}$):

Mahadik (2013) developed a control chart ‘ $\overline{X}$ chart with variable control and warning limits’ (*VCWL*-$\overline{X}$). A numerical comparison of the $\overline{X}$ chart and *VCWL*-$\overline{X}$ chart is carried out and it is observed that the *ARL* performance of the *VCWL*-$\overline{X}$ is efficient as compared to that of the $\overline{X}$ chart.

2.2. ‘Modified Control Chart with Warning Limits to detect shifts in the process mean’ ($MCCWL-\overline{X}$):

Rattihalli et al. (2021-2022) developed $MCCWL-\overline{X}$. This chart performs better as compared to the $\bar{X}$ chart. In $MCCWL-\overline{X}$, let A, W and R be the acceptance, warning and the rejection regions respectively. Corresponding to the observed sample of size $n_{mw\bar{x}}$, say, let ‘$\bar{X}$’ be the sample mean. In $MCCWL-\overline{X}$, it is assumed that, $\bar{X}_0$ (the sample mean at time zero) is in the acceptance region. Under the assumption of $MCCWL-\overline{X}$ mentioned above, for r = 1, 2, …, the stopping rule is ‘Declare the process as out of control if $\bar{X}_r \in R$ or $\bar{X}_r \in W$ and $\bar{X}_{(r+1)} \in W$’. Let $P_A = P\ (\bar{X} \in A)$. The terms P_W and P_R have the similar meaning. Under the assumption that, $\bar{X}_0 \in A$, the *ATS* expression of $MCCWL-\overline{X}$ is $\frac{n_{mw\bar{x}(1+P_W)}}{1-P_A(1+P_W)}$. They have numerically illustrated that, $MCCWL-\overline{X}$ performs better as compared to the $\overline{X}$ chart.

2.3. ‘Improved Control Chart to detect shift in the process mean’ (*ICC*-$\overline{X}$):

Gadre and Rattihalli (2022) developed *ICC*-$\overline{X}$. Let ‘*R*’ be the rejection region of *ICC*-$\overline{X}$. Stopping rule for this control chart is, ‘Stop if the sample mean of the first sample ($\overline{X}_1$, Say) falls in *R*; otherwise for *r* > 1, if $\overline{X}_r \in R$ and $\overline{X}_{r+1)} \in R$’. If it is assumed that $\overline{X}_0$ lie beyond the control limits, then this rule can be written as for r (*r* = 0, 1, …), ‘Declare the process as out of control if two successive sample means fall outside the control limits’. Assuming that, $\bar{X}_0 \in R$, the *ATS* expression of $ICC-\overline{X}$ is $\frac{n_{i\bar{x}}}{P_R^2}$. It is numerically illustrated that, *ICC*-$\overline{X}$ performs significantly better as compared to the $MCCWL-\overline{X}$ and the $\overline{X}$ chart.

In the next section, the notations, operation and *ATS* expression of the $IWL-\overline{X}$ chart is derived.

3. Basic Notations, the Operation, *ATS* criterion and Derivation of the *ATS* expression

Here it is assumed that the quality characteristic X has independent observations. Following are the notations to be used in $IWL - \overline{X}$ chart.

3.1. Basic Notations:

1. $\bar{X}$: Sample Mean
2. $n_{iw\bar{x}}$: Sample Size.
3. μ: Process mean.
4. μ_0: In-control process mean.
5. σ: The Process variability.
6. δ: Shift in the process mean.
7. δ_1: Design shift in the process mean, the magnitude of which is considered large enough to seriously impair the quality of the product.
8. μ_1: Out of control process mean. $\mu_1 = \mu_0 + \frac{\delta_1 \sigma}{\sqrt{n_{iw\bar{x}}}}$
9. $k_{iw\bar{x}}$: The coefficient of the control limits.
10. $k_{1iw\bar{x}}$: The coefficient of the warning limits.
11. $LCL_{iw\bar{x}} = \mu_0 - k_{iw\bar{x}} \frac{\sigma}{\sqrt{n_{iw\bar{x}}}} UCL_{iw\bar{x}} = \mu_0 + k_{iw\bar{x}} \frac{\sigma}{\sqrt{n_{iw\bar{x}}}}$ are the control limits.
12. $LWL_{iw\bar{x}} = \mu_0 - k_{1iw\bar{x}} \frac{\sigma}{\sqrt{n_{iw\bar{x}}}}$, $UWL_{iw\bar{x}} = \mu_0 + k_{1iw\bar{x}} \frac{\sigma}{\sqrt{n_{iw\bar{x}}}}$ are the warning limits.
13. $A = (LWL_{iw\bar{x}}, UWL_{iw\bar{x}})$, $W = (LCL_{iw\bar{x}}, LWL_{iw\bar{x}}) \cup (UWL_{iw\bar{x}}, UCL_{iw\bar{x}})$ and $R = (-\infty, LCL_{iw\bar{x}}) \cup (UCL_{iw\bar{x}}, \infty)$. Let '$S$' be the '*Signal'* and is the absorbing state.
14. A, W and R are the acceptance, warning and rejection regions respectively. Let 'S' be the '*Signal'* and is the absorbing state. Note that, $R \subset S$.
15. *ARL(δ)*: Average number of non-conforming samples inspected by the time the process has gone out of control to detect shift in the process mean from μ_0 to μ.
16. ARL_0 and ARL_1: In-control and out-of-control *ARLs*
17. *ATS(δ)*: Average number of non-conforming units inspected by the time the process has gone out of control to detect shift in the process mean from μ_0 to μ. Note that, *ATS(δ)*: $n_{iw\bar{x}}$(*ARL(δ)*). Also, ATS_0 and ATS_1: In-control and out-of-control *ATSs*.

18. τ: The minimum required value of ATS_0, the average time to signal when the process is in control.

3.2. The Operation, ATS criterion and the derivation of *ATS* expression of *IWL-$\bar{X}$* chart*:*

Let *CNT* be the counter. A flow chart of the implementation of *IWL-$\bar{X}$* chart is given below.

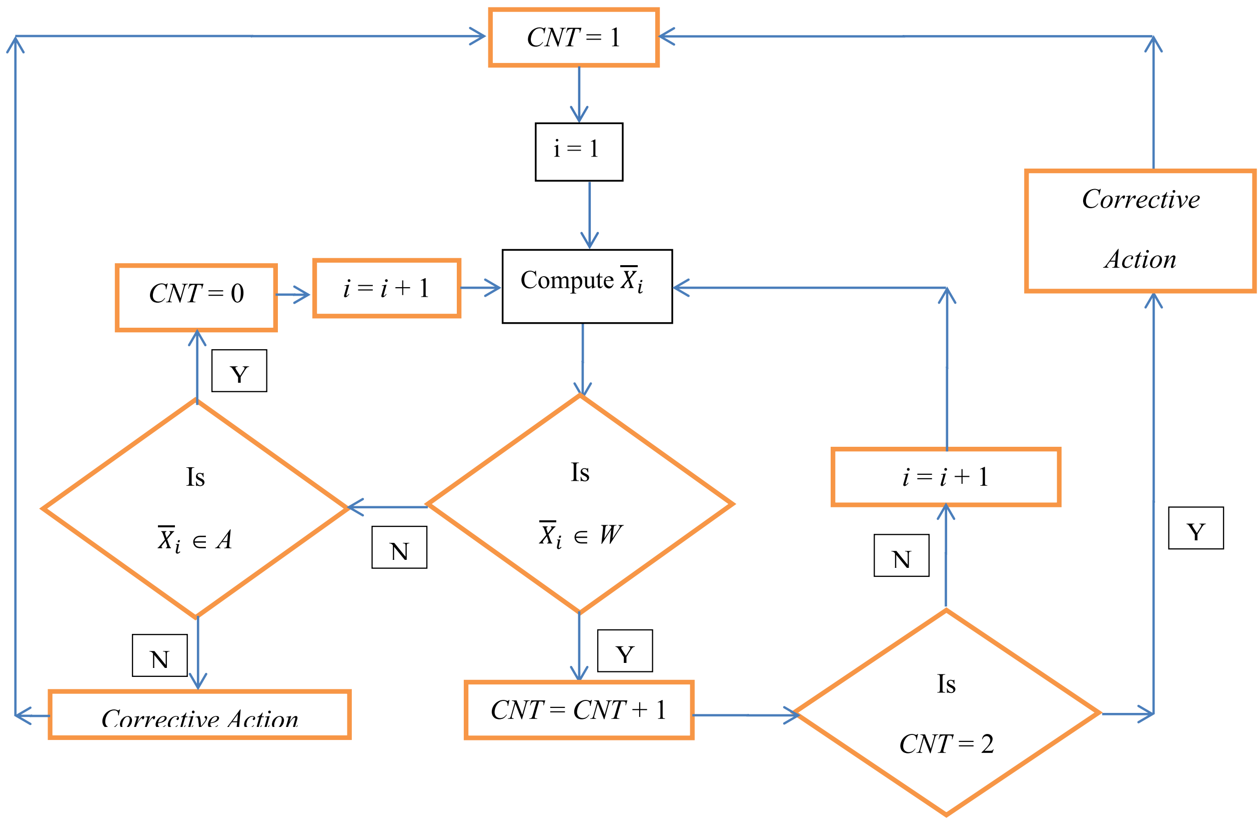

Figure 1. A Flow Chart of the implementation of *IWL-$\bar{X}$* Chart

Wu et al. (2001) used the *ATS* model. The design parameters of the *IWL-$\bar{X}$* chart are determined by using the *ATS* model, which is given as,

$$\left.\begin{array}{r} \text{Minimize } ATS_1 \\ \text{Subjectto the condition} \\ ATS_0 \geq \tau \end{array}\right\} \qquad (1)$$

To find the ATS expression of $IWL\text{-}\bar{X}$ chart, first derive its ARL expression. As mentioned in **Sub-Section (3.1),** A, W and R are the acceptance, warning and rejection regions and 'S' is the absorbing state respectively.

When the production process is of 'independent and identically distributed' (iid) random variables and the sample size $n_{iw\bar{x}}$ is fixed, then the sequence $\{\overline{X}_i\}_{i=1}^{\infty}$of the sample means are also iid. random variables. For $IWL - \overline{X}$ chart, it is enough to consider the sequence of three associated states $\{A, W, S\}$. Let $P_A = P(\overline{X} \in A)$, $P_W = P(\overline{X} \in W)$ and $P_R = P(\overline{X} \in R)$. To adapt Markov chain approach of Brook and Evans (1972, consider a current sample mean and the just previous sample mean (if such one exists). Following Markov chain approach of Brook and Evans (1972), with the state space $\{A, W, S\}$ the expression for ARL of the $IWL - \overline{X}$ chart can be obtained. The 'transition probability matrix' (t.p.m.) $\boldsymbol{P}$ is given by

$$\boldsymbol{P} = \begin{array}{c} \\ A \\ W \\ S \end{array} \begin{array}{c} \begin{array}{ccc} A & W & S \end{array} \\ \begin{pmatrix} P_A & P_W & P_R \\ P_A & 0 & P_W + P_R \\ 0 & 0 & 1 \end{pmatrix} \end{array} \qquad (2)$$

Let $\boldsymbol{P_1}$ be a matrix obtained by deleting the row and column of $\boldsymbol{P}$ corresponding to the signal state. $\underline{ARL}$ is $(\boldsymbol{I}-\boldsymbol{P_1})^{-1}\underline{\boldsymbol{1}}$, where $\underline{\boldsymbol{1}}$ is a column of unit elements. It is easy to verify that,

$$\underline{ARL} = \begin{bmatrix} \dfrac{1+P_W}{1-P_A(1+P_W)} \\ \dfrac{1}{1-P_A(1+P_W)} \end{bmatrix} \qquad (3)$$

Here, in $IWL - \overline{X}$ chart, it assumed that, at time zero, the value of $\overline{X} \in W$. With this assumption, from Brooke and Evans (1972) and as $ATS = n_{iw\bar{x}}(ARL)$, the ATS of this chart is

$$ATS = \frac{n_{iw\bar{x}}}{1 - P_A\langle 1 + P_W \rangle}. \qquad (4)$$

Now, as the quality characteristic $X \sim N(\mu, \sigma^2)$ distribution and the observations are independent. Note that $\bar{X} \sim N(\mu, \sigma^2/n)$. As mentioned in the notations of $IWL - \bar{X}$,

$P_W = P(\bar{X} \in W)$

$= P\{(\bar{X} \in (LCL_{iw\bar{x}}, LWL_{iw\bar{x}}))\, U\, (\bar{X} \in (UWL_{iw\bar{x}}, UCL_{iw\bar{x}}))\}$

$= P\left(\mu_0 + k_{1iw\bar{x}} \frac{\sigma}{\sqrt{n_{iw\bar{x}}}} < \bar{X} < \mu_0 + \frac{k_{iw\bar{x}}\sigma}{\sqrt{n_{iw\bar{x}}}}\right) + P\left(\mu_0 - k_{iw\bar{x}} \frac{\sigma}{\sqrt{n_{iw\bar{x}}}} < \bar{X} < \mu_0 - \frac{k_{1iw\bar{x}}\sigma}{\sqrt{n_{iw\bar{x}}}}\right)$

$= \varphi(k_{iw\bar{x}} - \delta\sqrt{n_{iw\bar{x}}}) - \varphi(k_{1iw\bar{x}} - \delta\sqrt{n_{iw\bar{x}}}) + \varphi(-k_{1iw\bar{x}} - \delta\sqrt{n_{iw\bar{x}}}) - \varphi(-k_{iw\bar{x}} - \delta\sqrt{n_{iw\bar{x}}})$

$=\{\varphi(k_{iw\bar{x}} - \delta\sqrt{n_{iw\bar{x}}}) - \varphi(-k_{iw\bar{x}} - \delta\sqrt{n_{iw\bar{x}}})\}-\{\varphi(k_{1iw\bar{x}} - \delta\sqrt{n_{iw\bar{x}}}) - \varphi(-k_{1iw\bar{x}} - \delta\sqrt{n_{iw\bar{x}}})\}$

Similarly, $P_A = P(\bar{X} \in A) = P(\bar{X} \in (LWL_{iw\bar{x}}, UWL_{iw\bar{x}}))$ and $P_R = P(\bar{X} \in R) = 1 - P_A - P_W$.

. Further, for given δ, $ATS(\delta)$ is a function of the design parameters $n_{iw\bar{x}}, k_{1iw\bar{x}}$ and $k_{iw\bar{x}}$. Using *ATS* model given in Equation (*1*), optimal values of the design parameters of $IWL - \bar{X}$ are obtained by using the *ATS* expression given in Equation (*4*). In the following, numerical illustrations are considered to compare the performance of $IWL - \bar{X}$ chart with the $\bar{X}$ chart, $MCCWL - \bar{X}$ and $ICC - \bar{X}$. A MAT-Lab code is developed to obtain the design parameters together with ATS_0 and ATS_1 values of the three charts.

4. Numerical Examples

Example 1:

Let $\mu_0 = 0$, $\sigma = 1$, $\delta_1 = 0.2$ and $\tau = 2000$. For these input parameters, values of the design parameters for the $\bar{X}$ chart, $MCCWL - \bar{X}$, $ICC - \bar{X}$ and $IWL - \bar{X}$ chart along with the respective ATS_1 values are given below.

(i) $\bar{X}$ chart: $n_{\bar{x}} = 111$, $k_{\bar{x}} = 1.9150$, $ATS_{1\bar{x}} = 192.6385$

(ii) $MCCWL - \bar{X}$: $n_{mw\bar{x}} = 107$, $k_{1mw\bar{x}} = 1.575$, $k_{mw\bar{x}} = 1.965$, $ATS_{1m\bar{x}} = 190.9142$

(iii) *ICC*- $\bar{X}$: $n_{i\bar{x}} = 97$, $k_{i\bar{x}} = 1.226$, $ATS_{1i\bar{x}} = 162.6760$

(iv) $IWL - \bar{X}$ chart: $n_{iw\bar{x}} = 91$, $k_{1iw\bar{x}} = 1.315$, $k_{iw\bar{x}} = 2.505$, $ATS_{1iw\bar{x}} = 151.6311$

This example shows that, not only $ATS_{1iw\bar{x}}$ for the process mean is less than $ATS_{1\bar{x}}$, $ATS_{1m\bar{x}}$ and $ATS_{1icc\bar{x}}$, but also the sample size $n_{iw\bar{x}}$ is not exceeding those of the related three charts, which is an indicative of reduction of the delay in detection of the shift.

Example 1 (Cont.):

For further comparison of $IWL-\bar{X}$ chart with the $\bar{X}$ chart, $MCCWL-\bar{X}$ and $ICC-\bar{X}$ for various values of δ, the *ATS (δ)* values are computed for these four charts by varying δ values from 0 to 0.3. These values are given in Table 1. Normalized *ATS* values (normalized w.r.t. the $\bar{X}$ chart) against δ values related to Table 1 are given in Figure 2.

Table 1. *ATSs* and the Normalized *ATSs* of the four charts for various δ values

δ Values	*ATS*				Normalized *ATS*			
	$\bar{X}$ Chart	$MCCWL-\bar{X}$	$ICC-\bar{X}$	$IWL-\bar{X}$ Chart	$\bar{X}$ Chart	$MCCWL-\bar{X}$	$ICC-\bar{X}$	$IWL-\bar{X}$ Chart
0	2000.3	2000.7	2000.5	2001.2	1	1.00019997	1.0001	1.000449933
0.01	1925.5	1953	1960.5	1955.7	1	1.014282005	1.018177	1.015684238
0.02	1821.8	1822.3	1848.0	1829.5	1	1.000274454	1.014381	1.004226589
0.03	1638.2	1638.4	1682.4	1648	1	1.000122085	1.026981	1.005982176
0.04	1434.5	1434	1488.1	1441	1	0.999651446	1.037365	1.004531196
0.05	1235.5	1233.9	1287.6	1233.8	1	0.998704978	1.042169	0.998624039
0.06	1055.1	1052.5	1097.4	1042.3	1	0.997535779	1.040091	0.987868448
0.07	898.8428	895.2351	926.9	874.5521	1	0.995986284	1.031215	0.972975586
0.08	766.9137	762.6142	779.9	732.4474	1	0.994393763	1.016933	0.955058437
0.09	657.1115	652.4132	656.4	614.6395	1	0.992850072	0.998917	0.935365611
0.10	566.34	561.512	554.5	518.22	1	0.991475086	0.979094	0.915033372
0.11	491.4603	486.7203	471.1	439.8501	1	0.990355274	0.958572	0.894986024
0.12	429.6475	425.1538	403.3	376.3338	1	0.989540961	0.938676	0.875912929
0.13	378.5014	374.3561	348.2	324.8639	1	0.989048125	0.919944	0.858289824
0.14	336.0429	332.3002	303.5	283.0876	1	0.988862434	0.903158	0.842415061
0.15	300.6659	297.3418	267.1	249.0864	1	0.988944207	0.888361	0.828449119
0.16	271.0766	268.1588	237.3	221.3208	1	0.989236253	0.875398	0.816451143
0.17	246.236	243.6921	213	198.566	1	0.989668854	0.865024	0.806405237
0.18	225.3092	223.0935	192.9	179.8517	1	0.990165959	0.856157	0.798243924
0.19	207.6243	205.6831	176.4	164.4097	1	0.99065042	0.849612	0.79186155
0.20	192.6385	190.9142	162.7	151.6313	1	0.991049037	0.844587	0.787128741
0.21	179.9112	178.3456	151.3	141.0322	1	0.991297929	0.84097	0.783898946
0.22	169.0836	167.6199	141.9	132.2256	1	0.991343335	0.83923	0.782013158
0.23	159.8616	158.4461	134	124.901	1	0.991145466	0.838225	0.781307081
0.24	152.0027	150.5863	127.4	118.807	1	0.990681744	0.838143	0.781611116
0.25	145.3059	143.8448	121.9	113.7392	1	0.989944662	0.83892	0.782756929
0.26	139.6036	138.0598	117.4	109.53	1	0.988941546	0.840953	0.784578621
0.27	134.7547	133.0966	113.6	106.0407	1	0.98769542	0.843013	0.786916523
0.28	130.6402	128.8425	110.4	103.1557	1	0.986239305	0.845069	0.789616825
0.29	127.1583	125.2021	107.8	100.7785	1	0.984616026	0.847762	0.792543625
0.3	124.2219	122.0946	105.7	98.8272	1	0.982875	0.850897	0.795569863

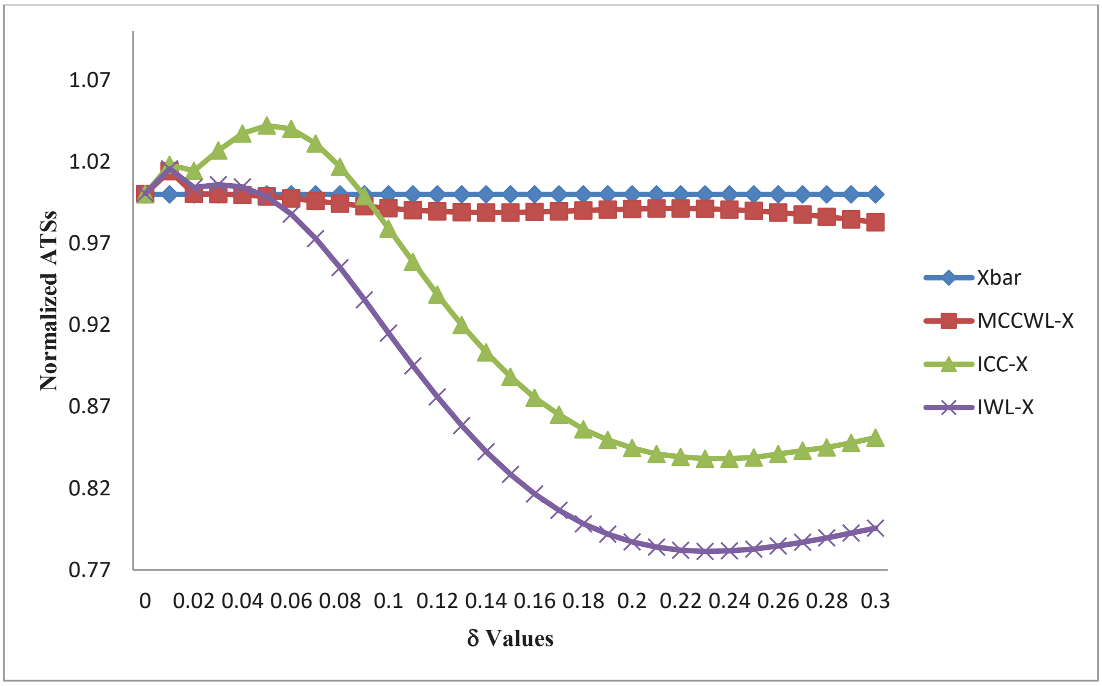

Figure 2. Normalized ATSs of the four charts

Result-1: From Figure 2, it is observed that, for $\delta > 0.04$, $IWL - \bar{X}$ chart signals faster than the related three control charts.

Example 2:

Here, to study the effect on the performance of the proposed $IWL - \bar{X}$ chart, 9 various combinations of input parameter values of (δ_1, τ) are considered. These 9 combinations of the input parameters are

δ_I: 0.2, 0.5, 1

τ: 2000, 10000, 50000

Considering all possible 9 combinations of the input parameters (δ_I, τ) values, the design parameters along with respective ATS_I values are computed for each of the four control charts and are given in **Table 2**.

Table 2. Design parameters of the $\bar{X}$ chart, $MCCWL-\bar{X}$, $ICC-\bar{X}$ and $IWL-\bar{X}$ chart with respective ATS_1 values

Ex	δ_1	τ	$\bar{X}$ Chart			$MCCWL-\bar{X}$				$ICC-\bar{X}$			$IWL-\bar{X}$ Chart			
			$n_{\bar{x}}$	$k_{\bar{x}}$	ATS_1	$n_{m\bar{x}}$	$k_{1m\bar{x}}$	$k_{m\bar{x}}$	ATS_1	$n_{i\bar{x}}$	$k_{i\bar{x}}$	ATS_1	$n_{iw\bar{x}}$	$k_{1iw\bar{x}}$	$k_{iw\bar{x}}$	ATS_1
1	0.2	2000	**112**	1.91103	**192.617**	**107**	1.575	1.962	**190.9142**	**97**	1.226	**162.6760**	**91**	1.315	2.505	**151.6313**
2	0.5	2000	**32**	2.40892	**48.2964**	**28**	1.905	2.51	**47.2533**	**24**	1.602	**37.3445**	**24**	1.65	3.03	**35.2685**
3	1	2000	**11**	2.77621	**15.590**	**10**	2.175	2.85	**15.1322**	**8**	1.858	**11.4993**	**8**	1.90	3.37	**10.9645**
4	0.2	10000	**186**	2.35344	**287.984**	**167**	1.835	2.455	**282.2001**	**148**	1.548	**224.4742**	**141**	1.615	2.955	**211.8496**
5	0.5	10000	**45**	2.84082	**64.6439**	**39**	2.165	2.95	**62.5997**	**33**	1.9	**47.3820**	**32**	1.955	3.4	**45.1721**
6	1	10000	**14**	3.19473	**19.78**	**12**	2.415	3.295	**19.0275**	**10**	2.15	**14.0284**	**10**	2.1840	3.7660	**13.4460**
7	0.2	50000	**269**	2.78335	**389.646**	**238**	2.14	2.88	**377.7186**	**196**	1.862	**287.3577**	**191**	1.914	3.3640	**273.6465**
8	0.5	50000	**59**	3.24362	**81.4096**	**51**	2.435	3.345	**78.2709**	**42**	2.184	**57.5073**	**40**	2.23	3.81	**55.1688**
9	1	50000	**18**	3.5681	**23.9998**	**15**	2.66	3.67	**22.9871**	**12**	2.422	**16.5576**	**12**	2.480	3.985	**15.9934**

Results: From Table 2, for all the 9 combinations, observe the following.

Result 2: $n_{iw\bar{x}} \leq n_{i\bar{x}} \leq n_{m\bar{x}} \leq n_{\bar{x}}$.

Result 3: $ATS_{1iw\bar{x}} \leq ATS_{1i\bar{x}} \leq ATS_{1m\bar{x}} \leq ATS_{1\bar{x}}$. This indicates that $IWL-\bar{X}$ chart detect shifts sooner. Thus, $IWL-\bar{X}$ chart performs better as compared to the related three charts.

Example 3:

Here various δ_1 values are as given in **Example 2** and small values of τ are 50, 100 and 150. Thus, the 9 combinations of (δ_1, τ) are as below.

δ_1: 0.2, 0.5, 1

τ: 50, 100, 150

Considering all possible 9 combinations of the input parameters (δ_1, τ), values of the design parameters along with respective values of ATS_1 are computed for each of the four control charts and are given in **Table 3**.

Table 3. Design parameters of the $\bar{X}$chart, $MCCWL - \bar{X}$, $ICC - \bar{X}$ and $IWL - \bar{X}$ chart with respective ATS_1 values (For small tau values)

Ex	δ_1	τ	$\bar{X}$ Chart			$MCCWL - \bar{X}$				$ICC - \bar{X}$			$IWL - \bar{X}$ Chart			
			$n_{\bar{x}}$	$k_{\bar{x}}$	ATS_1	$n_{m\bar{x}}$	$k_{1m\bar{x}}$	$k_{m\bar{x}}$	ATS_1	$n_{i\bar{x}}$	$k_{i\bar{x}}$	ATS_1	$n_{iw\bar{x}}$	$k_{1iw\bar{x}}$	$k_{iw\bar{x}}$	ATS_1
1	0.2	50	**12**	1.175	**34.7089**	**13**	1.0450	1.13	**34.6855**	**13**	0.6590	**35.1048**	**13**	0.79	1.505	**32.86**
2	0.5	50	**8**	1.41	**15.8714**	**8**	1.29	1.41	**15.8002**	**8**	0.8420	**15.0766**	**7**	0.995	1.915	**13.77**
3	1	50	**4**	1.755	**6.7018**	**4**	1.47	1.775	**6.6602**	**3**	1.1630	**5.8317**	**3**	1.24	2.375	**5.39**
4	0.2	100	**24**	1.175	**54.7688**	**23**	1.10	1.20	**54.7462**	**23**	0.7070	**54.8787**	**22**	0.845	1.635	**50.52**
5	0.5	100	**11**	1.60	**21.0000**	**11**	1.33	1.625	**20.9161**	**10**	1.0030	**19.1147**	**10**	1.10	2.095	**17.59**
6	1	100	**5**	1.96	**8.2133**	**5**	1.595	1.995	**8.1513**	**4**	1.2820	**6.8504**	**4**	1.35	2.56	**6.40**
7	0.2	150	**32**	1.245	**69.0387**	**30**	1.19	1.285	**68.9634**	**31**	0.7480	**68.2438**	**29**	0.88	1.75	**62.47**
8	0.5	150	**13**	1.715	**24.2904**	**13**	1.450	1.735	**24.1478**	**12**	1.0740	**21.4904**	**11**	1.19	2.255	**19.89**
9	1	150	**6**	2.055	**9.1828**	**5**	1.67	2.18	**9.0368**	**5**	1.3330	**7.4920**	**4**	1.70	2.705	**7.03**

Results: Looking to the entries in Table 3, for all 9 combinations, the following results are observed.

Result 4: $n_{iw\bar{x}} \leq n_{i\bar{x}} \leq n_{m\bar{x}} \leq n_{\bar{x}}$, except for $\delta_1 = 0.2$ and $\tau = 50$; and for $\delta_1 = 0.2$ and $\tau = 150$.

Result 5: For this combination, $n_{\bar{x}} < n_{iw\bar{x}} = n_{m\bar{x}}$. Also, $ATS_{1iw\bar{x}}$ is significantly smaller as compared to the related three control charts.

5. Conclusions

As given in Example 1, the proposed $IWL - \bar{X}$ chart performs better as compared to the $\bar{X}$ chart, $MCCWL - \bar{X}$, and $ICC - \bar{X}$. Also, for various $\delta > 0.04$ values, $ATS(\delta)$ of the $IWL - \bar{X}$ chart is significantly less as compared to the $\bar{X}$ chart, $MCCWL - \bar{X}$ and $ICC - \bar{X}$ chart. For combinations of the input parameters considered in Example 2 and in Example 3, $ATS_{1iw\bar{x}} \leq ATS_{1i\bar{x}} \leq ATS_{1m\bar{x}} \leq ATS_{1\bar{x}}$.

Implementation/operation of the rule to stop the process for the proposed chart is simple. As the proposed chart gives a scope to account for doubtful situations, and decision are taken by getting additional information, proposed chart seems to be used by the industries.

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